

THE MOST SYMMETRIC SURFACES IN THE 3-TORUS

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ABSTRACT. Suppose an orientation preserving action of a finite group G on the closed surface Σ_g of genus $g > 1$ extends over the 3-torus T^3 for some embedding $\Sigma_g \subset T^3$. Then $|G| \leq 12(g-1)$, and this upper bound $12(g-1)$ can be achieved for $g = n^2 + 1, 3n^2 + 1, 2n^3 + 1, 4n^3 + 1, 8n^3 + 1, n \in \mathbb{Z}_+$. Those surfaces in T^3 realizing the maximum symmetries can be either unknotted or knotted. Similar problems in non-orientable category is also discussed.

Connection with minimal surfaces in T^3 is addressed and when the maximum symmetric surfaces above can be realized by minimal surfaces is identified.

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1. INTRODUCTION

Let Σ_g be the closed orientable surface of genus $g > 1$, Π_g be the closed non-orientable surface of genus $g > 2$, and T^3 be the three dimensional torus (3-torus for short). We consider the following question in the smooth category:

Suppose the action of a finite group G on a closed surface S can extend over T^3 . Then what is the maximum order of the group and what does the maximum action look like?

A similar problem has been addressed for surfaces embedded in the 3-sphere, S^3 , which is the simplest compact 3-manifold in the sense that it is a one point compactification of our three space and the universal spherical 3-manifold covering all spherical 3-manifolds. See [WWZZ1] and [WWZZ2] for surfaces in the orientable category. Note that only orientable surfaces Σ_g

2010 *Mathematics Subject Classification.* 57M60, 57N10, 57S25, 53A10, 20F65, 05C10;
Key words and phrases. maximum surface symmetry in 3-torus, minimal surface.

The second author is supported by grant No.11501534 of NSFC and the last author is supported by grant No.11371034 of NSFC.

can be embedded in S^3 . T^3 is another natural and significant 3-manifold for this question. It is the universal compact Euclidean 3-manifold in the sense that it covers all compact Euclidean 3-manifolds; moreover T^3 is covered by our 3-space and the preimage of a closed surface $S \subset T^3$ under such a covering can be a triply periodic surface, which is an interesting object in the natural sciences and engineering [HBLL+].

Definition 1.1. Let G be a finite group. A G -action on a closed surface S is extendable over T^3 with respect to an embedding $e : S \hookrightarrow T^3$ if G can act on T^3 such that $h \circ e = e \circ h$ for any $h \in G$.

For an embedding of an orientable surface in T^3 we can define whether it is unknotted in the same way as the usual definition for knotting in S^3 .

Definition 1.2. An embedding $e : \Sigma_g \hookrightarrow T^3$ is unknotted if $e(\Sigma_g)$ splits T^3 into two handlebodies. Otherwise it is knotted.

We assume all orientable manifolds in this note are already oriented. According to whether the action preserves the orientation of the surface and T^3 , we can define four classes of maximum orders.

Definition 1.3. Let S be either Σ_g or Π_g . Define $E(S)$, $E^+(S)$, $E_+(\Sigma_g)$ and $E_+^+(\Sigma_g)$ as below:

$E(S)$: the maximum order of all extendable G -actions on S .

$E^+(S)$: the maximum order of all extendable G -actions on S which preserve the orientation of T^3 .

$E_+(\Sigma_g)$: the maximum order of all extendable G -actions on Σ_g which preserve the orientation of Σ_g .

$E_+^+(\Sigma_g)$: the maximum order of extendable G -actions on Σ_g , which preserve the orientation of both T^3 and Σ_g .

We will prove the following Theorem 1.4 and Theorem 1.5. Theorem 1.4 gives the upper bounds of those invariants defined in Definition 1.3, and Theorem 1.5 provides infinitely many g to realize each upper bound in Theorem 1.4.

Theorem 1.4. Suppose the genus $g > 1$ for Σ_g and $g > 2$ for Π_g .

- (1) $E_+^+(\Sigma_g) \leq 12(g-1)$.
- (2) $E_+(\Sigma_g) \leq 24(g-1)$, $E^+(\Sigma_g) \leq 24(g-1)$, $E(\Sigma_g) \leq 48(g-1)$.
- (3) $E^+(\Pi_g) \leq 12(g-2)$, $E(\Pi_g) \leq 24(g-2)$.

If an extendable G -action on S realizes one of the above upper bounds, then the corresponding surface $e(S)$ can be thought as a most symmetric surface in T^3 .

Theorem 1.5. Suppose n is any positive integer.

- (1) The upper bound of $E_+^+(\Sigma_g)$ in Theorem 1.4 can be achieved by an unknotted embedding for $g = 2n^3 + 1, 4n^3 + 1, 8n^3 + 1$; and by a knotted embedding for $g = 2n^3 + 1, 4n^3 + 1, 8n^3 + 1$, where 6 does not divide n ; and $g = n^2 + 1, 3n^2 + 1$.

(2) The upper bound of $E_+(\Sigma_g)$, $E^+(\Sigma_g)$ and $E(\Sigma_g)$ in Theorem 1.4 can be achieved for $g = 2n^3 + 1, 4n^3 + 1, 8n^3 + 1$ by an unknotted embedding.

(3) The upper bound of $E^+(\Pi_g)$ and $E(\Pi_g)$ in Theorem 1.4 can be achieved for $g = 2n^3 + 2, 8n^3 + 2$, where n is odd.

The proof of Theorem 1.4, using the Riemann-Hurwitz Formula and the Equivariant Loop Theorem, is given in §2. The proof of Theorem 1.5 occupies the remaining three sections of the paper, which is outlined as below:

In §3 and §4 we will construct various examples to realize those upper bounds in Theorem 1.5 (1), §3 for the unknotted case and §4 for the knotted case, therefore Theorem 1.5 (1) follows, up to the verification of the knottedness of those embeddings. Those examples are explicit in the following sense: The 3-torus is obtained by standard opposite face identification of the cube and we define those surfaces in the cube before the identification. Indeed, for each g in Theorem 1.5 (1), we construct all embeddings $\Sigma_g \subset T^3$ realizing $E_+^+(\Sigma_g)$ we can image for the moment, and we expect those are all embeddings realizing $E_+^+(\Sigma_g)$, see a conjecture below. The constructions in §3 and §4 often rely on an understanding of crystallographic space groups.

The knottedness of the examples in §3 and §4 involved in Theorem 1.5 (1), as well as Theorem 1.5 (2) and (3), could be verified by arguments such as those in §3 and §4, and by some topological reasoning; but we present a more convenient and interesting way in §5. In §5 we first link the verification of knottedness of the examples in §3 and §4 with minimal surfaces in T^3 , or equivalently, triply periodic minimal surfaces in R^3 , which itself is an important topic, see [Me] and [FH] for examples. Historically, many minimal surfaces in T^3 were constructed using the symmetry of T^3 and it is known that minimal surfaces in T^3 must be unknotted [Me]. We identify the examples in §3 with known triply periodic minimal surfaces (up to isotopy) [Br], therefore they are unknotted. On the other hand, by a simple criterion from covering space theory and our constructions, the examples in §4 are knotted, therefore can not be realized by minimal surfaces. We also identify the group actions in §3 and §4 with known space groups as well as their index-2 supergroups; then Theorem 1.5 (2) and (3) are proved.

Corollary 1.6. $\Sigma_g \subset T^3$ realizing the upper bound $E_+^+(g) = 12(g - 1)$ can be realized by minimal surfaces for $g = 2n^3 + 1, 4n^3 + 1, 8n^3 + 1$.

For each g it is easy to construct an extendable action of order $4(g - 1)$ on (T^3, Σ_g) , see Example 2.3. We end the introduction by the following

Conjecture 1.7. Suppose the genus $g > 1$.

(1) All g realizing the upper bound $E_+^+(\Sigma_g) = 12(g - 1)$ are listed in Theorem 1.5 (1), and moreover, all examples realizing those g are listed in §3 and §4.

(2) $E_+^+(\Sigma_g) = 4(g - 1)$ for $g \in Z_+ \setminus K$, where $K = \{f_i(n) | n \in Z_+\}$ and f_i runs over finitely many quadratic and cubic functions.

2. UPPER AND LOWER BOUNDS OF EXTENDABLE FINITE GROUPS

We need the following two important results, see [Hu], [MY], to prove Theorem 1.4.

Riemann-Hurwitz Formula. $\Sigma_g \rightarrow \Sigma_{g'}$ is a regular branched covering with transformation group G . Let a_1, a_2, \dots, a_k be the branched points in $\Sigma_{g'}$ having indices $q_1 \leq q_2 \leq \dots \leq q_k$. Then

$$2 - 2g = |G|(2 - 2g' - \sum_{i=1}^k (1 - \frac{1}{q_i}))$$

Equivariant Loop Theorem. Let M be a three manifold with a smooth action of a discrete group G . Let F be an equivariant subsurface of ∂M . If F is not π_1 -injective with respect to inclusion into M , then it admits a G -equivariant compression disk.

Here a nonempty subset X of M is G -equivariant if $h(X) = X$ or $h(X) \cap X = \emptyset$, for any $h \in G$. A disk $D \subset M$ is a compression disk of F if $\partial D \subset F$ and in F it does not bound any disk.

Proof of (1). Suppose there is an extendable G -action on Σ_g which preserves the orientation of both T^3 and Σ_g . Cutting T^3 along $e(\Sigma_g)$ we get a three manifold M . Since both Σ_g and T^3 are orientable, Σ_g must be two-sided in T^3 , therefore M contains two boundaries F_1 and F_2 , and $F_1 \cong F_2 \cong \Sigma_g$.

Now clearly G acts on M . Since the G -action preserves the orientation of both T^3 and Σ_g , each of F_1 and F_2 is a G -equivariant surface. Since $g > 1$, $\pi_1(\Sigma_g)$ is not abelian, but $\pi_1(T^3)$ is abelian so the induced homomorphism $\pi_1(\Sigma_g) \rightarrow \pi_1(T^3)$ is not injective. Then at least one of F_1 and F_2 is not π_1 -injective with respect to inclusion into M . Suppose F_1 is not π_1 -injective, then it admits a G -equivariant compression disk D of F_1 by the Equivariant Loop Theorem.

If there exists an $h \in G$ such that $h(D) = D$ and h reverses an orientation of D , then we can choose a G -equivariant regular neighbourhood $N(D)$ of D such that there is a homeomorphism $i : D \times [-1, 1] \rightarrow N(D)$ and $i(D \times \{0\}) = D$. Then $D' = i(D \times \{1\})$ is also a G -equivariant compression disk of F_1 , and clearly for this D' , if $h'(D') = D'$ for some $h' \in G$, then h' preserves an orientation of D' . Hence we can assume each element of G that preserves D also preserves its orientation, in particular it is fixed point free on ∂D .

Since the G -action on T^3 preserves the orientations of both Σ_g and T^3 , the induced G -action on M preserves F_1 . With the quotient topology F_1/G is homeomorphic to some $\Sigma_{g'}$, and $p : F_1 \rightarrow F_1/G$ is a regular branched covering. Since the G action is fixed point free on ∂D by previous discussion, $p(\partial D)$ is a simple closed curve in F_1/G .

Let a_1, a_2, \dots, a_k be the branch points having indices $q_1 \leq q_2 \leq \dots \leq q_k$. Note $2 - 2g < 0$. By the Riemann-Hurwitz Formula we have

$$2 - 2g' - \sum_{i=1}^k \left(1 - \frac{1}{q_i}\right) = \frac{2 - 2g}{|G|} < 0.$$

If $g' = 0$ and $k \leq 3$, then $p(\partial D)$ must bound a disk in F_1/G containing at most one branch point, and ∂D will bound a disk in F_1 , a contradiction. Hence either $g' \geq 1$ or $g' = 0$ and $k \geq 4$.

Notice that for each q_i , we have

$$1 - \frac{1}{q_i} \geq 1/2.$$

Then by elementary calculation we have $|G| \leq 12(g-1)$, and moreover the equality holds if and only if $g' = 0$, $k = 4$ and $(q_1, q_2, q_3, q_4) = (2, 2, 2, 3)$. \square

Remark 2.1. Zimmermann first proved the order of orientation preserving finite group action on handlebody of genus g is bounded by $12(g-1)$ [Zi] soon after the work [MY].

By the above proof, we actually have the following:

Theorem 2.2. *Suppose Σ_g is embedded in a three manifold M and a finite group G acts on (Σ_g, M) . If G preserves both the two sides and the orientation of Σ_g and Σ_g is not π_1 -injective in M , then $|G| \leq 12(g-1)$.*

Now (2) and (3) follow from (1).

Proof of (2). Suppose there is an extendable G -action on Σ_g . Let G° be the normal subgroup of G containing all elements that preserve both the orientations of Σ_g and T^3 . In the case of $E_+(\Sigma_g)$ or $E^+(\Sigma_g)$, the index of G° in G is at most two, and in the case of $E(\Sigma_g)$, the index of G° in G is at most four. Hence $E_+(\Sigma_g) \leq 24(g-1)$, $E^+(\Sigma_g) \leq 24(g-1)$, $E(\Sigma_g) \leq 48(g-1)$. \square

Proof of (3). Suppose there is an extendable G -action on Π_g and the action preserves the orientation of T^3 . We can choose an equivariant regular neighbourhood $N(\Pi_g)$ of Π_g , then G also acts on $\partial N(\Pi_g)$. Since T^3 is orientable, $N(\Pi_g)$ is homeomorphic to a twisted $[-1, 1]$ -bundle over Π_g and $\partial N(\Pi_g)$ is the orientable double cover of Π_g under the bundle projection. Since $\chi(\Pi_g) = 2 - g$, $\chi(\partial N(\Pi_g)) = 4 - 2g = 2 - 2(g-1)$. It follows that $\partial N(\Pi_g)$ is homeomorphic to Σ_{g-1} . Clearly the G -action preserves the two sides of $\partial N(\Pi_g)$. Since the action preserves the orientation of T^3 , it also preserves the orientation of $\partial N(\Pi_g)$. Then by the result of (1), we have $E^+(\Pi_g) \leq 12(g-2)$. And similar to the proof of (2), we have $E(\Pi_g) \leq 24(g-2)$. \square

Example 2.3. Let $([0, 1]^3, +)$ be the unit cube with a cross properly embedded shown as the right side of Figure 1. For each $g > 1$, put $g-1$ copies of $([0, 1]^3, +)$ to get $([0, 1]^3, +)_{g-1}$ which is a cuboid with an antennae properly embedded, shown as Figure 1. If we identify the opposite faces of the cuboid,

we get (T^3, Θ_g) , where Θ_g is a graph of genus g . If we first only identify the right and left faces, we get $([0, 1]^2 \times S^1, A_{g-1})$, where A_{g-1} is a circular antennae with $g - 1$ bars. It is easy to see that $G = D_{g-1} \oplus Z_2$ acts on the pair $([0, 1]^2 \times S^1, A_{g-1})$, where D_{g-1} is the dihedral group acts in a standard way on $([0, 1]^2 \times S^1, A_{g-1})$, and Z_2 is π -rotation of $([0, 1]^2 \times S^1, A_{g-1})$ around the center-circle. Clear this action preserves the each pair of faces of $([0, 1]^2 \times S^1, A_{g-1})$ to be identified, therefore induces an action on (T^3, Θ_g) . Let $N(\Theta_g)$ be a G -invariant regular neighborhood of Θ_g . Then $\partial N(\Theta_g) = \Sigma_g$ and G of order $4(g - 1)$ acts on (T^3, Σ_g) .

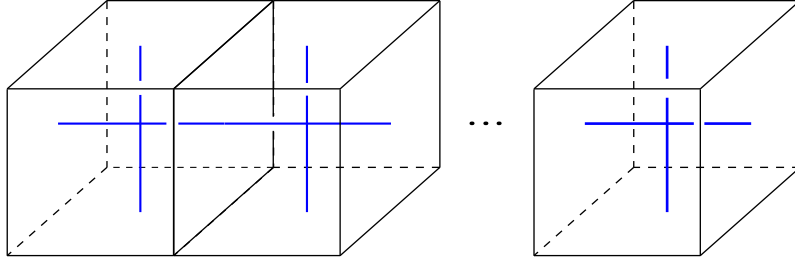


FIGURE 1.

3. UNKNOTTED EXAMPLES OF THE MOST SYMMETRIC SURFACES

In this section we give three classes of examples realizing the upper bound of $E_+^+(\Sigma_g) = 12(g - 1)$. In each case we first construct a triply periodic graph Γ in the three-dimensional Euclidean space E^3 . There will be a three-dimensional space group \mathcal{G} preserving Γ . Then we choose a rank-three translation normal subgroup T in \mathcal{G} . The space E^3/T is our T^3 . The finite group $G = \mathcal{G}/T$ acts on T^3 preserving the graph $\Theta = \Gamma/T$. Finally, we choose an equivariant regular neighbourhood $N(\Theta)$ of Θ , and $\partial N(\Theta)$ is our surface Σ_g . After the discussion of §5 we will see that for each example the complement of $N(\Theta)$ is also a handlebody, hence the surface is unknotted.

Definition 3.1. Let $T_1 = \{(a, b, c) \mid a, b, c \in \mathbb{Z}\}$ be the group of integer translations. Let $t_x = (1, 0, 0)$, $t_y = (0, 1, 0)$, $t_z = (0, 0, 1)$. An element $t = (a, b, c) \in T_1$ acts on E^3 as following:

$$t : (x, y, z) \mapsto (x + a, y + b, z + c)$$

For $n \in \mathbb{Z}_+$, we define three classes of subgroups T_m of T_1 for those integers m that can be presented (uniquely) in the form n^3 , $2n^3$ and $4n^3$ as follows:

$$T_{n^3} = \langle nt_x, nt_y, nt_z \rangle,$$

$$T_{2n^3} = \langle nt_y + nt_z, nt_z + nt_x, nt_x + nt_y \rangle,$$

$$T_{4n^3} = \langle -nt_x + nt_y + nt_z, nt_x - nt_y + nt_z, nt_x + nt_y - nt_z \rangle.$$

Here the subscript m of T_m is equal to the volume $\text{Vol}(E^3/T_m)$, $m = n^3, 2n^3, 4n^3$.

Clearly T_{2n^3} and T_{4n^3} are subgroups of T_{n^3} . One can easily verify that $T_{(2n)^3} \subset T_{2n^3}$, since $2nt_x, 2nt_y, 2nt_z$ are linear combinations of $nt_y + nt_z, nt_z + nt_x, nt_x + nt_y$, hence $T_{32n^3} = T_{4(2n)^3} \subset T_{(2n)^3} = T_{8n^3} \subset T_{2n^3}$. Similarly one can verify that $T_{(2n)^3} \subset T_{4n^3}$, hence $T_{16n^3} = T_{2(2n)^3} \subset T_{(2n)^3} = T_{8n^3} \subset T_{4n^3}$.

Definition 3.2. We define five isometries of E^3 as follows:

$$\begin{aligned} r_y &: (x, y, z) \mapsto (-x, y, -z) \\ r_z &: (x, y, z) \mapsto (-x, -y, z) \\ r_{xy} &: (x, y, z) \mapsto (y, x, -z) \\ r_{xyz} &: (x, y, z) \mapsto (z, x, y) \\ t_{1/2} &: (x, y, z) \mapsto (x + 1/2, y + 1/2, z + 1/2) \end{aligned}$$

The isometries r_y , r_z and r_{xy} are 2-fold rotations (i.e., rotation by an angle of π) about the y -axis, z -axis and the line $x = y, z = 0$ respectively. The isometry r_{xyz} is a positive 3-fold rotation about the cube body diagonal, $x = y = z$ (i.e., right-hand rule rotation by $2\pi/3$ about the direction $[1, 1, 1]$.)

The notations of space groups used below come from [Ha].

Example 3.3. Let Γ_{min}^1 be the one skeleton of the unit cube $[0, 1]^3$ in E^3 as in Figure 2. Let $\Gamma^1 = \bigcup_{t \in T_1} t(\Gamma_{min}^1)$. Then Γ^1 , the 1-skeleton of the tessellation of E^3 by the unit cube, is a triply periodic graph called the simple or primitive cubic lattice (**pcu** in [RCSR]).

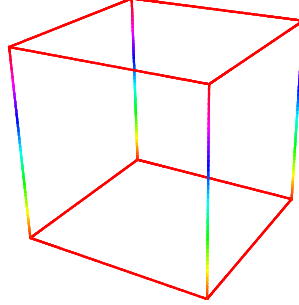


FIGURE 2. One skeleton of $[0, 1]^3$

Let $H^1 = \langle r_y, r_z, r_{xy}, r_{xyz} \rangle$, which is the well-known orientation-preserving isometric group of $[-1, 1]^3$ of order 24 (with Schönflies symbol O). Let $\mathcal{G}^1 = \langle T_1, H^1 \rangle$, this is the space group $P432$. Then \mathcal{G}^1 preserves Γ^1 . Now in $T^3 \cong E^3/T_1$ we have a graph $\Theta_1^1 = \Gamma^1/T_1$. It has one vertex and three edges, hence its Euler characteristic $\chi(\Theta_1^1) = -2$ and its genus $g = 1 - \chi(\Theta_1^1) = 3$. Clearly $G_1^1 = \mathcal{G}^1/T_1 \cong H^1$ acts on $T^3 \cong E^3/T_1$ preserving Θ_1^1 , and G_1^1 has order 24. Hence when we choose an equivariant regular neighbourhood $N(\Theta_1^1)$ of Θ_1^1 , we get an extendable action of order 24 on $\Sigma_3 \cong \partial N(\Theta_1^1)$.

Similarly in $T^3 \cong E^3/T_{n^3}$ we have a graph $\Theta_{n^3}^1 = \Gamma^1/T_{n^3}$. Since T_{n^3} is a normal subgroup of \mathcal{G}^1 , $G_{n^3}^1 = \mathcal{G}^1/T_{n^3}$ acts on $T^3 \cong E^3/T_{n^3}$ preserving $\Theta_{n^3}^1$.

$$\begin{aligned}\chi(\Theta_{n^3}^1) &= \chi(\Theta_1^1) \cdot \text{Vol}(E^3/T_{n^3}) = -2n^3 \\ |G_{n^3}^1| &= |G_1^1| \cdot \text{Vol}(E^3/T_{n^3}) = 24n^3\end{aligned}$$

Hence the genus of $\Theta_{n^3}^1$ is $2n^3 + 1$. Then we can get an extendable action of order $24n^3$ on $\Sigma_{2n^3+1} \cong \partial N(\Theta_{n^3}^1)$.

Notice that when $m = 2n^3, 4n^3, n \in \mathbb{Z}_+$, T_m is also a normal subgroup of \mathcal{G}^1 . Similar to the above construction, we can get an extendable action of order $24m$ on $\Sigma_{2m+1} \cong \partial N(\Theta_m^1)$. Here $\Theta_m^1 = \Gamma^1/T_m$ is a graph in $T^3 \cong E^3/T_m$.

In the above example the superscript 1 in Γ^1 or Θ_m^1 is equal to the volume of the ‘minimal 3-torus’ E^3/T_1 . It is also equal to the number of vertices of the graph Θ_1^1 . In following examples we use similar notations.

Example 3.4. Let Γ_{min}^2 be the graph in the unit cube $[0, 1]^3$ in E^3 as in Figure 3. It consists of four edges from $(1/2, 1/2, 1/2)$ to $(0, 0, 0)$, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$. Let $\Gamma^2 = \bigcup_{t \in T_2} t(\Gamma_{min}^2)$. One can check that it is connected and is known to scientists as the bonding structure of diamond (**dia** in [RCSR]). Figure 3 shows a fundamental region of T_2 .

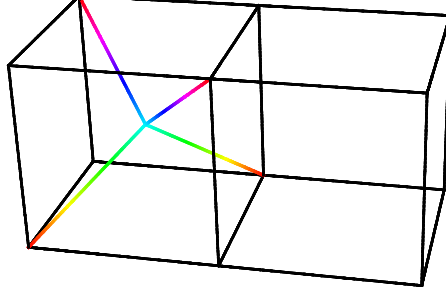


FIGURE 3. Γ_{min}^2 in $[0, 2] \times [0, 1] \times [0, 1]$

Let $H^2 = \langle r_y, r_z, r_{xyz} \rangle$. It is the orientation-preserving isometric group of the regular tetrahedron formed by the convex hull of $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, and $(-1, -1, 1)$ and has Schönflies symbol T . Let $\mathcal{G}^2 = \langle T_2, H^2, t_{1/2} r_{xy} \rangle$, this is the space group $F4_132$. Then \mathcal{G}^2 preserves Γ^2 .

Now in $T^3 \cong E^3/T_2$ we have a graph $\Theta_2^2 = \Gamma^2/T_2$. It has two vertices and four edges, hence its Euler characteristic $\chi(\Theta_2^2) = -2$ and its genus is 3. T_2 is a normal subgroup of \mathcal{G}^2 and $G_2^2 = \mathcal{G}^2/T_2$ has order 24. It acts on $T^3 \cong E^3/T_2$ preserving Θ_2^2 . Hence when we choose an equivariant regular neighbourhood $N(\Theta_2^2)$ of Θ_2^2 , we get an extendable action of order 24 on $\Sigma_3 \cong \partial N(\Theta_2^2)$.

Similarly when $m = n^3, 4n^3, 16n^3, n \in \mathbb{Z}_+$, in $T^3 \cong E^3/T_{2m}$ we have a graph $\Theta_{2m}^2 = \Gamma^2/T_{2m}$. One can check that T_{2m} is a normal subgroup of \mathcal{G}^2 ,

and the quotient $G_{2m}^2 = \mathcal{G}^2/T_{2m}$ acts on $T^3 \cong E^3/T_{2m}$ preserving Θ_{2m}^2 .

$$\chi(\Theta_{2m}^2) = \chi(\Theta_2^2) \cdot \text{Vol}(E^3/T_{2m})/\text{Vol}(E^3/T_2) = -2m$$

$$|G_{2m}^2| = |G_2^2| \cdot \text{Vol}(E^3/T_{2m})/\text{Vol}(E^3/T_2) = 24m$$

Hence the genus of Θ_{2m}^2 is $2m + 1$. Then we can get an extendable action of order $24m$ on $\Sigma_{2m+1} \cong \partial N(\Theta_{2m}^2)$.

Example 3.5. Let γ be the graph in the unit cube $[0, 1]^3$ in E^3 as in Figure 4. It has three edges from $(1/4, 1/4, 1/4)$ to $(0, 1/2, 1/4)$, $(1/4, 0, 1/2)$ and $(1/2, 1/4, 0)$. Let $\Gamma_{min}^4 = \gamma \cup t_y^2 t_z r_y r_z(\gamma) \cup t_x^2 t_y t_z r_y(\gamma) \cup t_x^2 t_y r_z(\gamma)$. Then let $\Gamma^4 = \bigcup_{t \in T_4} t(\Gamma_{min}^4)$. One can check that it is connected, and is the oft-rediscovered chiral vertex-transitive net of degree-3 known by many names [HOP], including **srs** in [RCSR]. Figure 4 shows a fundamental region of T_4 .

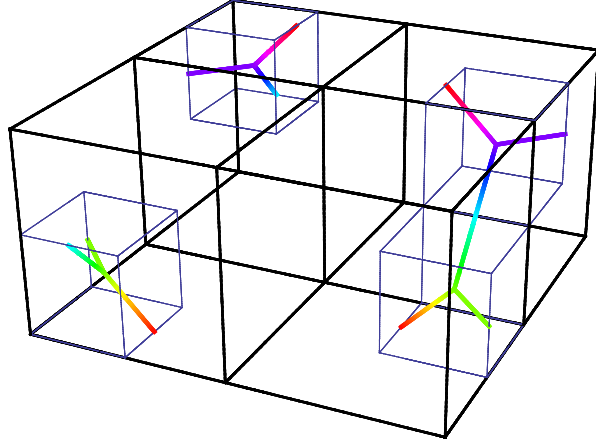


FIGURE 4. Γ_{min}^4 in $[0, 2] \times [0, 2] \times [0, 1]$

Let $\mathcal{G}^4 = \langle T_4, t_x t_z r_z, t_y t_z r_y, r_{xyz}, t_x t_{1/2} r_{xy} \rangle$, it is the space group $I4_132$. Then \mathcal{G}^4 preserves Γ^4 .

Now in $T^3 \cong E^3/T_4$ we have a graph $\Theta_4^4 = \Gamma^4/T_4$. It has four vertices and six edges, hence its Euler characteristic $\chi(\Theta_4^4) = -2$ and its genus is 3. T_4 is a normal subgroup of \mathcal{G}^4 and $G_4^4 = \mathcal{G}^4/T_4$ has order 24. It acts on $T^3 \cong E^3/T_4$ preserving Θ_4^4 . Hence when we choose an equivariant regular neighbourhood $N(\Theta_4^4)$ of Θ_4^4 , we get an extendable action of order 24 on $\Sigma_3 \cong \partial N(\Theta_4^4)$.

Similarly when $m = n^3, 2n^3, 4n^3, n \in \mathbb{Z}_+$, in $T^3 \cong E^3/T_{4m}$ we have a graph $\Theta_{4m}^4 = \Gamma^4/T_{4m}$. One can check that T_{4m} is a normal subgroup of \mathcal{G}^4 , and the quotient $G_{4m}^4 = \mathcal{G}^4/T_{4m}$ acts on $T^3 \cong E^3/T_{4m}$ preserving Θ_{4m}^4 .

$$\chi(\Theta_{4m}^4) = \chi(\Theta_4^4) \cdot \text{Vol}(E^3/T_{4m})/\text{Vol}(E^3/T_4) = -2m$$

$$|G_{4m}^4| = |G_4^4| \cdot \text{Vol}(E^3/T_{4m})/\text{Vol}(E^3/T_4) = 24m$$

Hence the genus of Θ_{4m}^4 is $2m + 1$. Then we can get an extendable action of order $24m$ on $\Sigma_{2m+1} \cong \partial N(\Theta_{4m}^4)$.

4. KNOTTED EXAMPLES OF THE MOST SYMMETRIC SURFACES

In the section we give a further six classes of examples realizing the upper bound of $E_+^+(\Sigma_g) = 12(g - 1)$. In each case below the embedding is knotted. The surface $e(\Sigma_g)$ does bound a handlebody in T^3 on one side, but the complement of the handlebody is not a handlebody, and this will be clear after the discussion of §5.

Constructions of the extendable actions and surfaces is similar to §4, but here the graph Γ in E^3 can be disconnected. There is a space group \mathcal{G} preserving Γ and a rank three translation normal subgroup T in \mathcal{G} so that the graph $\Theta = \Gamma/T$ is connected. The finite group $G = \mathcal{G}/T$ acts on $T^3 \cong E^3/T$ preserving Θ . Then we choose an equivariant regular neighbourhood $N(\Theta)$ of Θ , and $\partial N(\Theta)$ is our surface Σ_g .

Example 4.1. Let $\Gamma^{1,2} = \Gamma^1 \cup t_{1/2}(\Gamma^1)$. The graph $t_{1/2}(\Gamma^1)$ can be thought of as the dual of Γ^1 in E^3 . $\Gamma^{1,2}$ has two connected components and is named **pcu-c** in [RCSR]. In the unit cube $[0, 1]^3$ it is as in Figure 5.

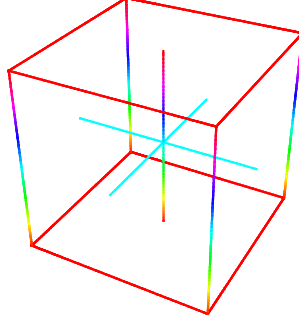


FIGURE 5. Part of $\Gamma^{1,2}$ in $[0, 1]^3$

Let $\mathcal{G}^{1,2} = \langle \mathcal{G}^1, t_{1/2} \rangle$, it is the space group $I432$, and let $T_{1/2} = \langle T_1, t_{1/2} \rangle = \{(a/2, b/2, c/2) \mid (a, b, c) \in T_4\}$. Then $\mathcal{G}^{1,2}$ preserves $\Gamma^{1,2}$ and $T_{1/2}$ is a normal subgroup of $\mathcal{G}^{1,2}$. Since $t_{1/2}$ changes the two components of $\Gamma^{1,2}$, the graph $\Theta_{1/2}^{1,2} = \Gamma^{1,2}/T_{1/2}$ in $T^3 \cong E^3/T_{1/2}$ is connected. Its Euler characteristic $\chi(\Theta_{1/2}^{1,2}) = -2$ and its genus is 3. $G_{1/2}^{1,2} = \mathcal{G}^{1,2}/T_{1/2}$ has order 24. It acts on $T^3 \cong E^3/T_{1/2}$ preserving $\Theta_{1/2}^{1,2}$. Hence when we choose an equivariant regular neighbourhood $N(\Theta_{1/2}^{1,2})$ of $\Theta_{1/2}^{1,2}$, we get an extendable action of order 24 on $\Sigma_3 \cong \partial N(\Theta_{1/2}^{1,2})$.

Similarly let $T_{n^3/2} = \{(a/2, b/2, c/2) \mid (a, b, c) \in T_{4n^3}\}, n \in \mathbb{Z}_+, 2 \nmid n$. Then the graph $\Theta_{n^3/2}^{1,2} = \Gamma^{1,2}/T_{n^3/2}$ in $T^3 \cong E^3/T_{n^3/2}$ is connected. Since

$T_{n^3/2}$ is a normal subgroup of $\mathcal{G}^{1,2}$, $G_{n^3/2}^{1,2} = \mathcal{G}^{1,2}/T_{n^3/2}$ acts on $T^3 \cong E^3/T_{n^3/2}$ preserving $\Theta_{n^3/2}^{1,2}$.

$$\begin{aligned}\chi(\Theta_{n^3/2}^{1,2}) &= \chi(\Theta_{1/2}^{1,2}) \cdot \text{Vol}(E^3/T_{n^3/2})/\text{Vol}(E^3/T_{1/2}) = -2n^3 \\ |G_{n^3/2}^{1,2}| &= |G_{1/2}^{1,2}| \cdot \text{Vol}(E^3/T_{n^3/2})/\text{Vol}(E^3/T_{1/2}) = 24n^3\end{aligned}$$

Hence the genus of $\Theta_{n^3/2}^{1,2}$ is $2n^3 + 1$. Then we can get an extendable action of order $24n^3$ on $\Sigma_{2n^3+1} \cong \partial N(\Theta_{n^3/2}^{1,2})$, here $n \in \mathbb{Z}_+$, $2 \nmid n$.

Example 4.2. Let $\Gamma^{2,2} = \Gamma^2 \cup t_x(\Gamma^2)$. The graph $t_x(\Gamma^2)$ can be thought as the dual of Γ^2 in E^3 . $\Gamma^{2,2}$ has two connected components and is called **dia-c** in [RCSR]. In the fundamental region of T_2 it is as in Figure 6.

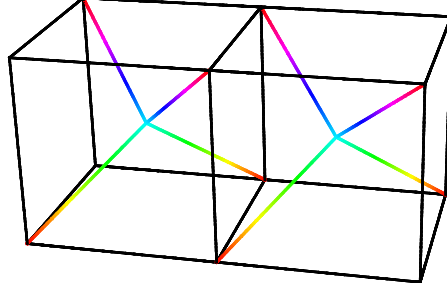


FIGURE 6. Part of $\Gamma^{2,2}$ in $[0, 2] \times [0, 1] \times [0, 1]$

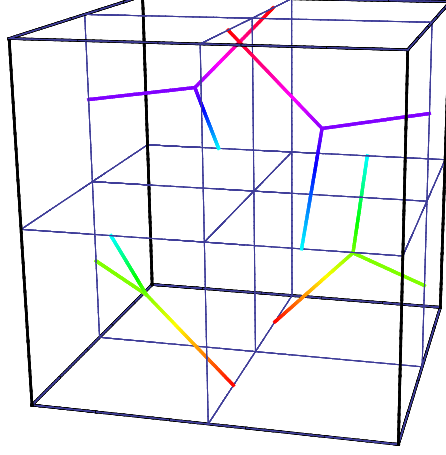
Let $\mathcal{G}^{2,2} = \langle \mathcal{G}^2, t_x \rangle$, it is the space group $P4_232$. Then $\mathcal{G}^{2,2}$ preserves $\Gamma^{2,2}$ and contains T_1 as a normal subgroup. Since t_x changes the two components of $\Gamma^{2,2}$, the graph $\Theta_1^{2,2} = \Gamma^{2,2}/T_1$ in $T^3 \cong E^3/T_1$ is connected. Its Euler characteristic $\chi(\Theta_1^{2,2}) = -2$, and its genus is 3. $G_1^{2,2} = \mathcal{G}^{2,2}/T_1$ has order 24. It acts on $T^3 \cong E^3/T_1$ preserving $\Theta_1^{2,2}$. Hence when we choose an equivariant regular neighbourhood $N(\Theta_1^{2,2})$ of $\Theta_1^{2,2}$, we get an extendable action of order 24 on $\Sigma_3 \cong \partial N(\Theta_1^{2,2})$.

Similarly if $m = n^3, 4n^3, n \in \mathbb{Z}_+, 2 \nmid n$, the graph $\Theta_m^{2,2} = \Gamma^{2,2}/T_m$ in $T^3 \cong E^3/T_m$ is connected. One can check that T_m is a normal subgroup of $\mathcal{G}^{2,2}$, $G_m^{2,2} = \mathcal{G}^{2,2}/T_m$ acts on $T^3 \cong E^3/T_m$ preserving $\Theta_m^{2,2}$.

$$\begin{aligned}\chi(\Theta_m^{2,2}) &= \chi(\Theta_1^{2,2}) \cdot \text{Vol}(E^3/T_m) = -2m \\ |G_m^{2,2}| &= |G_1^{2,2}| \cdot \text{Vol}(E^3/T_m) = 24m\end{aligned}$$

Hence the genus of $\Theta_m^{2,2}$ is $2m + 1$. Then we can get an extendable action of order $24m$ on $\Sigma_{2m+1} \cong \partial N(\Theta_m^{2,2})$, here $m = n^3, 4n^3, n \in \mathbb{Z}_+, 2 \nmid n$.

Example 4.3. Let $\Gamma^{4,4} = \Gamma^4 \cup t_x(\Gamma^4) \cup t_y(\Gamma^4) \cup t_z(\Gamma^4)$. $\Gamma^{4,4}$ has four connected components. In the unit cube $[0, 1]^3$ in E^3 it is as in Figure 7.

FIGURE 7. Part of $\Gamma^{4,4}$ in $[0, 1]^3$

One can check that $\mathcal{G}^{2,2}$ defined in Example 4.2 (space group $P4_232$) preserves $\Gamma^{4,4}$, and the graph $\Theta_1^{4,4} = \Gamma^{4,4}/T_1$ in $T^3 \cong E^3/T_1$ is connected. Its Euler characteristic $\chi(\Theta_1^{4,4}) = -2$, and its genus is 3. $G_1^{2,2} = \mathcal{G}^{2,2}/T_1$ has order 24. It acts on $T^3 \cong E^3/T_1$ preserving $\Theta_1^{4,4}$. Hence when we choose an equivariant regular neighbourhood $N(\Theta_1^{4,4})$ of $\Theta_1^{4,4}$, we get an extendable action of order 24 on $\Sigma_3 \cong \partial N(\Theta_1^{4,4})$.

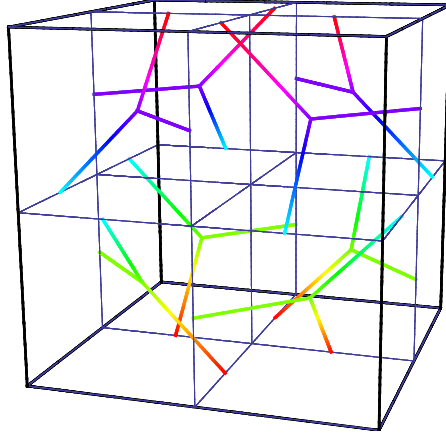
Similarly if $m = n^3, 2n^3, n \in \mathbb{Z}_+, 2 \nmid n$, the graph $\Theta_m^{4,4} = \Gamma^{4,4}/T_m$ in $T^3 \cong E^3/T_m$ is connected. $G_m^{2,2}$ acts on $T^3 \cong E^3/T_m$ preserving $\Theta_m^{4,4}$.

$$\begin{aligned}\chi(\Theta_m^{4,4}) &= \chi(\Theta_1^{4,4}) \cdot \text{Vol}(E^3/T_m) = -2m \\ |G_m^{2,2}| &= |G_1^{2,2}| \cdot \text{Vol}(E^3/T_m) = 24m\end{aligned}$$

Hence the genus of $\Theta_m^{4,4}$ is $2m + 1$. Then we can get an extendable action of order $24m$ on $\Sigma_{2m+1} \cong \partial N(\Theta_m^{4,4})$, here $m = n^3, 2n^3, n \in \mathbb{Z}_+, 2 \nmid n$.

Example 4.4. Let $\Gamma^{4,8} = \Gamma^{4,4} \cup t_{1/2}(\Gamma^{4,4})$. It has eight connected components. In the unit cube $[0, 1]^3$ it is as in Figure 8.

One can check that $\mathcal{G}^{1,2}$ defined in Example 4.1 (space group $I432$) preserves $\Gamma^{4,8}$, and the graph $\Theta_{1/2}^{4,8} = \Gamma^{4,8}/T_{1/2}$ in $T^3 \cong E^3/T_{1/2}$ is connected. Its Euler characteristic is -2 and its genus is 3. The order 24 group $G_{1/2}^{1,2}$ acts on $T^3 \cong E^3/T_{1/2}$ preserving $\Theta_{1/2}^{4,8}$. Hence when we choose an equivariant regular neighbourhood $N(\Theta_{1/2}^{4,8})$ of $\Theta_{1/2}^{4,8}$, we get an extendable action of order 24 on $\Sigma_3 \cong \partial N(\Theta_{1/2}^{4,8})$.

FIGURE 8. Part of $\Gamma^{4,8}$ in $[0, 1]^3$

Similarly for $n \in \mathbb{Z}_+, 2 \nmid n$, the graph $\Theta_{n^3/2}^{4,8} = \Gamma^{4,8}/T_{n^3/2}$ in $T^3 \cong E^3/T_{n^3/2}$ is connected. $G_{n^3/2}^{1,2}$ acts on $T^3 \cong E^3/T_{n^3/2}$ preserving $\Theta_{n^3/2}^{4,8}$.

$$\begin{aligned}\chi(\Theta_{n^3/2}^{4,8}) &= \chi(\Theta_{1/2}^{4,8}) \cdot \text{Vol}(E^3/T_{n^3/2})/\text{Vol}(E^3/T_{1/2}) = -2n^3 \\ |G_{n^3/2}^{1,2}| &= |G_{1/2}^{1,2}| \cdot \text{Vol}(E^3/T_{n^3/2})/\text{Vol}(E^3/T_{1/2}) = 24n^3\end{aligned}$$

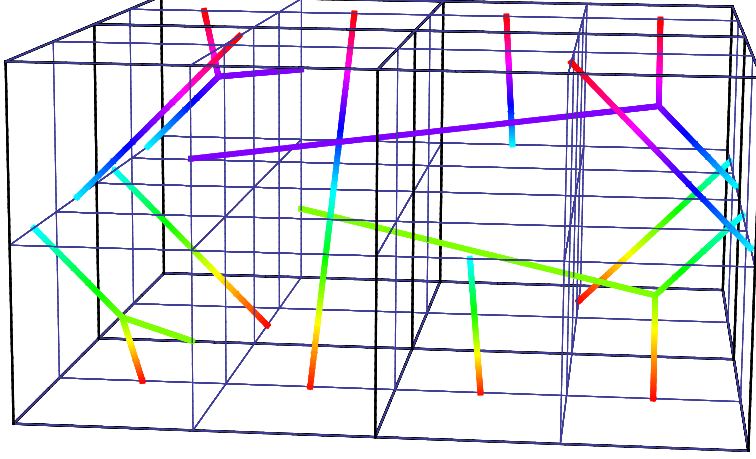
Hence the genus of $\Theta_{n^3/2}^{4,8}$ is $2n^3 + 1$. Then we can get an extendable action of order $24n^3$ on $\Sigma_{2n^3+1} \cong \partial N(\Theta_{n^3/2}^{4,8})$, here $n \in \mathbb{Z}_+, 2 \nmid n$.

Example 4.5. Let γ' be the graph in the unit cube $[0, 1]^3$ in E^3 as in Figure 3. There are three edges from $(1/4, 1/4, 1/4)$ to $(0, 1/4, 1/2)$, $(1/2, 0, 1/4)$, $(1/4, 1/2, 0)$, and three edges from $(0, 3/4, 1/2)$ to $(1/2, 3/4, 1)$, $(1/2, 0, 3/4)$ to $(1, 1/2, 3/4)$ and $(3/4, 1/2, 0)$ to $(3/4, 1, 1/2)$ separately.

Let $\Gamma_{min}^{4'} = \gamma' \cup t_y^2 t_z r_y r_z(\gamma') \cup t_x^2 t_y t_z r_y(\gamma') \cup t_x^2 t_y r_z(\gamma')$, and $\Gamma^4 = \bigcup_{t \in T_4} t(\Gamma_{min}^{4'})$. One can check that Γ^4 has 27 connected components, each is similar to a mirror image of Γ^4 . Figure 9 shows the part of Γ^4 in a fundamental region of T_4 .

By construction, \mathcal{G}^4 (space group $I4_132$) preserves Γ^4 . And one can check that $\Theta_4^4 = \Gamma^4/T_4$ in $T^3 \cong E^3/T_4$ is connected. Θ_4^4 has four vertices and six edges, hence it has Euler characteristic -2 and genus 3. The order-24 group G_4^4 acts on $T^3 \cong E^3/T_4$ preserving Θ_4^4 . Hence when we choose an equivariant regular neighbourhood $N(\Theta_4^4)$ of Θ_4^4 , we get an extendable action of order 24 on $\Sigma_3 \cong \partial N(\Theta_4^4)$.

Similarly when $m = n^3, 2n^3, 4n^3, n \in \mathbb{Z}_+, 3 \nmid n$, the graph $\Theta_{4m}^4 = \Gamma^4/T_{4m}$ in $T^3 \cong E^3/T_{4m}$ is connected. The quotient $G_{4m}^4 = \mathcal{G}^4/T_{4m}$ acts on $T^3 \cong$

FIGURE 9. Part of Γ'^4_{min} in $[0, 2] \times [0, 2] \times [0, 1]$

E^3/T_{4m} preserving Θ_{4m}^4 .

$$\begin{aligned}\chi(\Theta_{4m}^4) &= \chi(\Theta_4^4) \cdot \text{Vol}(E^3/T_{4m})/\text{Vol}(E^3/T_4) = -2m \\ |G_{4m}^4| &= |G_4^4| \cdot \text{Vol}(E^3/T_{4m})/\text{Vol}(E^3/T_4) = 24m\end{aligned}$$

Hence the genus of Θ_{4m}^4 is $2m + 1$. Then we can get an extendable action of order $24m$ on $\Sigma_{2m+1} \cong \partial N(\Theta_{4m}^4)$, here $m = n^3, 2n^3, 4n^3, n \in \mathbb{Z}_+, 3 \nmid n$.

Definition 4.6. Let $t_\omega = (-1/2, \sqrt{3}/2, 0)$. Define

$$\begin{aligned}T_{n^2}^\omega &= \langle nt_\omega, nt_x, t_z \rangle, \\ T_{3n^2}^\omega &= \langle 2nt_\omega + nt_x, nt_\omega + 2nt_x, t_z \rangle.\end{aligned}$$

Define a rotation r_ω on E^3 as following:

$$r_\omega : (x, y, z) \mapsto \left(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x - \frac{1}{2}y, z\right)$$

T_1^ω is the hexagonal 3D lattice and r_ω is a rotation by $2\pi/3$ right-handed with respect to the direction $[0, 0, 1]$.

Example 4.7. Let P be a regular hexagon in the xy -plane. It has center $(0, 0, 0)$ and contains $(1/2, \sqrt{3}/6, 0)$ as a vertex, see Figure 10. Let Γ_{min}^P be the boundary of P in E^3 . Let $\Gamma^P = \bigcup_{t \in T_1^\omega} t(\Gamma_{min}^P)$. It contains infinitely many connected components, and for each $n \in \mathbb{Z}$, the horizontal plane $\{(x, y, n) \mid x, y \in E^2\}$ contains exactly one of them. A local picture of Γ^P is as in Figure 11.

Let $H^\omega = \langle r_\omega, r_x, r_y \rangle$; it is the isometric group of P (Schönflies symbol D_6). Let $\mathcal{G}^\omega = \langle T_1^\omega, H^\omega \rangle$, this is the space group $P622$. Then \mathcal{G}^ω preserves Γ^P , and $\Theta_1^P = \Gamma^P/T_1^\omega$ is a connected graph in $T^3 \cong E^3/T_1^\omega$. It has two vertices and three edges, hence its Euler characteristic $\chi(\Theta_1^\omega) = -1$ and its genus is 2. Clearly $G_1^\omega = \mathcal{G}^\omega/T_1^\omega$ acts on $T^3 \cong E^3/T_1^\omega$ preserving Θ_1^P ,

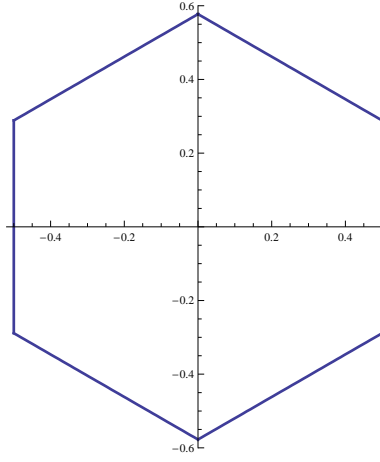
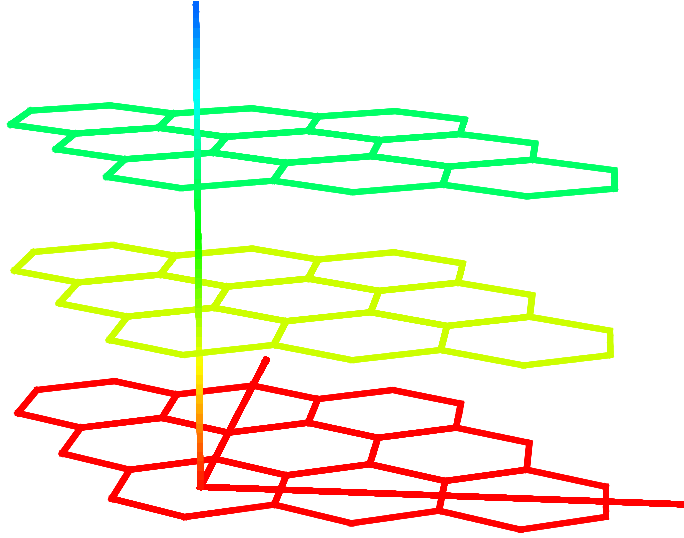


FIGURE 10. A regular hexagon

FIGURE 11. A local picture of Γ^P

and $G_1^\omega \cong H^\omega$ has order 12. Hence when we choose an equivariant regular neighbourhood $N(\Theta_1^P)$ of Θ_1^P , we get an extendable action of order 12 on $\Sigma_2 \cong \partial N(\Theta_1^P)$.

Notice that when $m = n^2, 3n^2, n \in \mathbb{Z}_+$, T_m^ω is a normal subgroup of \mathcal{G}^ω . Similarly the graph $\Theta_m^P = \Gamma^P / T_m^\omega$ in $T^3 \cong E^3 / T_m^\omega$ is connected, and $G_m^\omega = \mathcal{G}^\omega / T_m^\omega$ acts on $T^3 \cong E^3 / T_m^\omega$ preserving Θ_m^P .

$$\begin{aligned}\chi(\Theta_m^P) &= \chi(\Theta_1^P) \cdot \text{Vol}(E^3 / T_m^\omega) / \text{Vol}(E^3 / T_1^\omega) = -m \\ |G_m^\omega| &= |G_1^\omega| \cdot \text{Vol}(E^3 / T_m^\omega) / \text{Vol}(E^3 / T_1^\omega) = 12m\end{aligned}$$

Hence the genus of Θ_m^P is $m + 1$. Then we can get an extendable action of order $12m$ on $\Sigma_{m+1} \cong \partial N(\Theta_m^P)$.

5. MINIMAL SURFACES, SPACE GROUPS, PROOF OF THE MAIN RESULT

In this section we will finish the proof of Theorem 1.5. We need some results about triply periodic minimal surfaces, space groups, and a lemma given below.

There are three classical triply periodic minimal surfaces that admit high symmetries: Schwarz's P surface, Schwarz's D [Schw] and Schoen's gyroid surface illustrated in Figure 12 [Br]. Denote them by S_P , S_D and S_G . The fundamental region for S_D is a cube of volume 2, we redraw S_D in the fundamental region $[0, 1] \times [0, 2] \times [0, 1]$ we used.

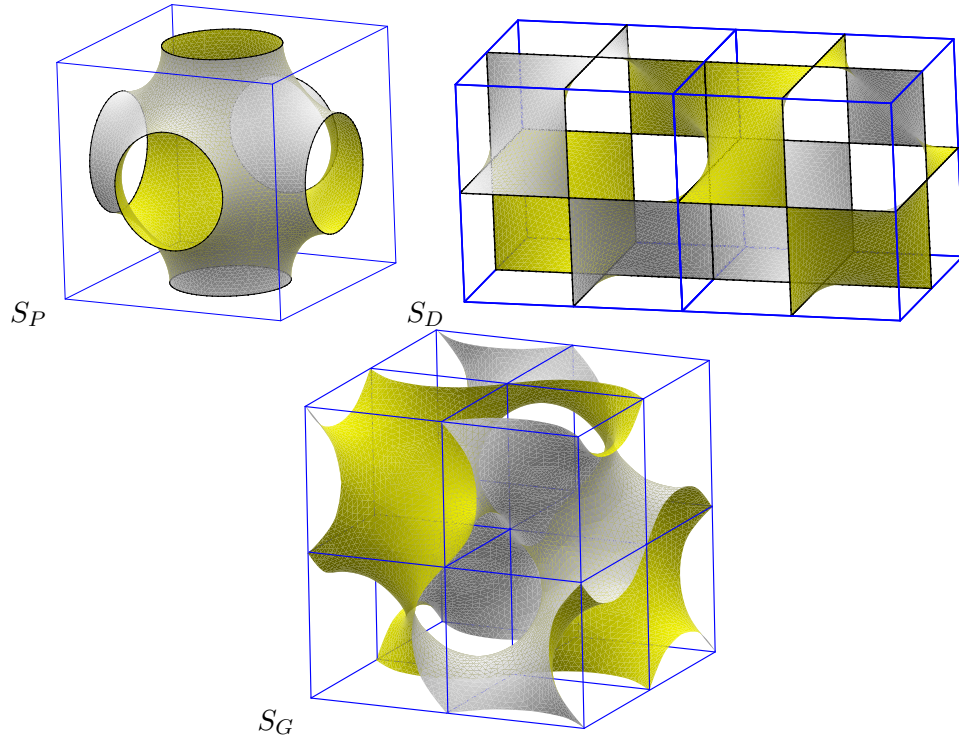


FIGURE 12. Translational unit cells of the minimal surfaces S_P , and S_D match the translational unit cells of their associated graphs, Γ^1 , Γ^2 , and are displayed with the same viewing direction. The picture of S_G above shows the cube $[0, 2] \times [0, 2] \times [0, 2]$, twice the region of that used to depict Γ^4 , but depicted with the same viewing direction.

The following results are known (and can be checked):

- (a) Boundaries of equivariant regular neighbourhoods of Γ^1 , Γ^2 and Γ^4 in Example 3.3, 3.4 and 3.5 can be realized by S_P , S_D and S_G respectively.

(b) S_P , S_D and S_G are unknotted in T^3 . Actually, any genus $g > 1$ minimal orientable closed surface in T^3 must be unknotted, [Me].

The notations of space groups below come from [Ha].

(c) \mathcal{G}^1 , \mathcal{G}^2 , \mathcal{G}^4 , $\mathcal{G}^{1,2}$ and $\mathcal{G}^{2,2}$ in Example 3.3, 3.4, 3.5, 4.1 and 4.2 are the space groups $[P432]$, $[F4_132]$, $[I4_132]$, $[I432]$ and $[P4_232]$ respectively.

(d) S_P , S_D , S_G are preserved by space groups $[Im\bar{3}m]$, $[Pn\bar{3}m]$, $[Ia\bar{3}d]$ respectively.

(e) We have the following index two subgroup sequences:

- $[P432] \subset [I432] \subset [Im\bar{3}m]$
- $[F4_132] \subset [P4_232] \subset [Pn\bar{3}m]$
- $[I4_132] \subset [Ia\bar{3}d]$

Lemma 5.1. *Let T be a lattice in E^3 , $p : E^3 \rightarrow T^3 = E^3/T$ be the covering map. Σ_g is an embedded surface in T^3 . If $p^{-1}(\Sigma_g)$ is not connected, then Σ_g is knotted.*

Proof. Note first, from the definition of unknotted embedding $\Sigma_g \subset T^3$, the induced map on the fundamental groups must be surjective. Let F be a connected component of $p^{-1}(\Sigma_g)$. Let $St(F)$ be its stable subgroup in T , i.e., the lattice translations that preserve F . If $p^{-1}(\Sigma_g)$ has more than one component then $St(F) \neq T$, hence $\pi_1(\Sigma_g) \rightarrow \pi_1(T^3)$ is not surjective. Hence Σ_g is knotted in T^3 . \square

Proof of Theorem 1.5. By the above results (a) and (b), the surfaces in T^3 described in §3 are unknotted. Since $\Gamma^{1,2}$, $\Gamma^{2,2}$, $\Gamma^{4,4}$, $\Gamma^{4,8}$, Γ'^4 and Γ^P are disconnected, by Lemma 5.1, the surfaces in T^3 described in §4 are knotted. Then by combining the Examples in §3 and §4, we finish the proof of Theorem 1.5 (1).

Now we are going to prove Theorem 1.5 (2) and (3) based on the examples in §3 and §4, and the results (c), (d) and (e) above.

For (2). Recall $\mathcal{G}^1 = [P432]$. Example 3.3 told us that when $m = n^3, 2n^3, 4n^3$ we have the action \mathcal{G}^1/T_m on Γ^1/T_m , or equivalently on S_P/T_m by (a), of order $24m$. By (d) $[Im\bar{3}m]$ preserves S_P . By (e), we have an order $96m$ extendable $[Im\bar{3}m]/T_m$ -action on $S_P/T_m \cong \Sigma_{2m+1}$ realizing the upper bound of $E(\Sigma_g)$, where $g = 2m + 1$.

The group $[Im\bar{3}m]/T_m$ contains two elements h_1 and h_2 satisfying: h_1 preserves the orientation of S_P/T_m and reverses the orientation of E^3/T_m , h_2 reverses the orientation of S_P/T_m and preserves the orientation of E^3/T_m . Then choosing the index two subgroups of $[Im\bar{3}m]$, $\langle [P432], h_1 \rangle$ and $\langle [P432], h_2 \rangle$, we get an extendable action on Σ_{2m+1} , realizing the upper bound of $E_+(\Sigma_g)$ and $E^+(\Sigma_g)$ respectively. Now (2) is proved.

For (3). Recall $\mathcal{G}^2 = [F4_132]$. Example 3.4 told us that when $m = n^3, 4n^3, 16n^3$ we have the action \mathcal{G}^2/T_{2m} on Γ^2/T_{2m} , or equivalently on S_D/T_{2m} by (a), of order $24m$.

Similarly we also have an order $96m$ extendable $[Pn\bar{3}m]/T_{2m}$ -action on $S_D/T_{2m} \cong \Sigma_{2m+1}$ realizing the upper bound of $E(\Sigma_g)$, where $g = 2m + 1$.

Furthermore if $m = n^3, 4n^3, n \in \mathbb{Z}_+, n$ odd, $[Pn\bar{3}m]/T_{2m}$ contains a translation h reversing an orientation of S_D/T_{2m} and preserving an orientation of E^3/T_{2m} (see Example 4.2 for details, or better to see S_D in Figure 12). Modulo this translation we can get an order $48m$ extendable $[Pn\bar{3}m]/T_m$ -action on $S_D/T_m \cong \Pi_{2m+2}$, realizing the upper bound $24(g-2)$ of $E(\Pi_g)$, where $g = 2m + 2$. Then choosing the index two subgroup $[P4_232] = \mathcal{G}^{2,2}$, we can get an extendable action on Π_{2m+2} , realizing the upper bound of $E^+(\Pi_g)$. Now (3) is proved. \square

Remark 5.2. In the above discussion for (3), the regular neighbourhood $N(S_D/T_m)$ of S_D/T_m is homeomorphic to a twisted $[-1, 1]$ -bundle over S_D/T_m . Its complement in E^3/T_m is essentially the regular neighbourhood $N(\Theta_m^{2,2})$ of $\Theta_m^{2,2}$ in Example 4.2. Similarly for $n \in \mathbb{Z}_+, n$ odd, and $m = n^3/2$, we can get an order $48m$ extendable $[Im\bar{3}m]/T_{n^3/2}$ -action on $S_P/T_{n^3/2} \cong \Pi_{2n^3+2}$, realizing the upper bound of $E(\Pi_g)$. Choosing the index two subgroup $[I432]$, we can get an extendable action on Π_{2n^3+2} , realizing the upper bound of $E^+(\Pi_g)$. The regular neighbourhood $N(S_P/T_{n^3/2})$ is homeomorphic to a twisted $[-1, 1]$ -bundle over $S_P/T_{n^3/2}$. Its complement in $E^3/T_{n^3/2}$ is essentially the regular neighbourhood $N(\Theta_{n^3/2}^{1,2})$ in Example 4.1. There is no similar construction for a regular neighbourhood of the Gyroid minimal surface because there is no translation that reverses the orientation of S_G/T_4 ; the handlebodies on each side of S_G are mirror images of one another.

REFERENCES

- [Br] K. Brakke, *Minimal surface page*,
<http://www.susqu.edu/brakke/evolver/examples/periodic/periodic.html>
- [FH] C. Frohman, J. Hass, *Unstable minimal surfaces and Heegaard splittings*. Invent. Math. 95 (1989), no. 3, 529-540.
- [Ha] T. Hahn (Ed.), *International tables for crystallography*, D. Reidel Publishing Company, (2005).
- [He] J. Hempel, *3-manifolds*, Princeton University Press, (1976).
- [Hu] A. Hurwitz, *Über algebraische Gebilde mit eindeutigen Transformationen in sich*, Mathematische Annalen, 41(3) (1892), 403-442.
- [HBLL+] S.T. Hyde, Z. Blum, T. Landh, S. Lidin, B.W. Ninham, S. Andersson and K. Larsson. *The Language of Shape: The Role of Curvature in Condensed Matter*. New York, USA: Elsevier, (1996).
- [HOP] S.T. Hyde, M. O’Keeffe, D.M. Proserpio, *A Short History of an Elusive Yet Ubiquitous Structure in Chemistry, Materials, and Mathematics*. Angewandte Chemie Int. Ed. 47, (2008), 79968000.
- [Me] W. H. Meeks, *The conformal structure and geometry of triply periodic minimal surfaces in R^3* , Bull. Amer. Math. Soc. 83 (1977), no. 1, 134-136.
- [MY] W. H. Meeks, S. T. Yau, *The equivariant Dehn’s lemma and loop theorem*, Commentarii Mathematici Helvetici, 56(1) (1981), 225-239.
- [RCSR] M. O’Keeffe, M.A. Peskov, S.J. Ramsden, O.M. Yaghi, *The Reticular Chemistry Structure Resource (RCSR) Database of, and Symbols for Crystal Nets* Accts. Chem. Res. 41, (2008), 1782-1789.

- [Schw] H. A. Schwarz, *Gesammelte Mathematische Abhandlungen*. Bd 1, Berlin: Springer, (1890).
- [Scho] A. H. Schoen, NASA Tech. Note No. D-5541, (1970)
- [WWZZ1] C. Wang, S. C. Wang, Y. M. Zhang, B. Zimmermann, *Extending finite group actions on surfaces over S^3* . *Topology Appl.* **160**, (2013), no. 16, 2088-2103.
- [WWZZ2] C. Wang, S. C. Wang, Y. M. Zhang, B. Zimmerman, *Maximal Orders of Extendable Finite Group Actions on Surfaces*. *Groups Geom. Dyn.* 9 (2015), no. 4, 1001-1045.
- [Zi] B. Zimmermann, *Über Homomorphismen n -dimensionaler Henkelkörper und endliche Erweiterungen von Schottky-Gruppen*, *Comment. Math. Helv.* 56 (1981), 474-486.

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